Problem 4.30

An electron is in the spin state

$$\chi = A \begin{pmatrix} 3i \\ 4 \end{pmatrix}$$

(a) Determine the normalization constant A.

(b) Find the expectation values of S_x, S_y, and S_z.

(c) Find the "uncertainties" σ_{S_x}, σ_{S_y}, and σ_{S_z}. *Note:* These sigmas are standard deviations, not Pauli matrices!

(d) Confirm that your results are consistent with all three uncertainty principles (Equation 4.100 and its cyclic permutations—only with S in place of L, of course).

Solution

Part (a)

Determine A by requiring the spin state to be normalized.

$$\chi = \begin{bmatrix} 3iA \\ 4A \end{bmatrix} \quad \Rightarrow \quad |3iA|^2 + |4A|^2 = 1$$

$$9A^2 + 16A^2 = 1$$

$$25A^2 = 1$$

$$A^2 = \frac{1}{25}$$

$$A = \pm\frac{1}{5}$$

Either sign works. Choose the plus sign.

$$A = \frac{1}{5}$$

Therefore, the spin state is

$$\chi = \frac{1}{5} \begin{bmatrix} 3i \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{3i}{5} \\ \frac{4}{5} \end{bmatrix}.$$

Part (b)

Calculate the expectation value of S_x.

$$\langle S_x \rangle = \frac{\langle \chi \,|\, S_x \,|\, \chi \rangle}{\langle \chi \,|\, \chi \rangle} = \frac{\chi^\dagger S_x \chi}{\chi^\dagger \chi} = \frac{\begin{bmatrix} \frac{3i}{5} \\ \frac{4}{5} \end{bmatrix}^\dagger \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{3i}{5} \\ \frac{4}{5} \end{bmatrix}}{\begin{bmatrix} \frac{3i}{5} \\ \frac{4}{5} \end{bmatrix}^\dagger \begin{bmatrix} \frac{3i}{5} \\ \frac{4}{5} \end{bmatrix}} = \frac{\hbar}{2} \frac{\begin{bmatrix} -\frac{3i}{5} & \frac{4}{5} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{3i}{5} \\ \frac{4}{5} \end{bmatrix}}{\begin{bmatrix} -\frac{3i}{5} & \frac{4}{5} \end{bmatrix} \begin{bmatrix} \frac{3i}{5} \\ \frac{4}{5} \end{bmatrix}}$$

$$= \frac{\hbar}{2} \frac{\begin{bmatrix} -\frac{3i}{5} & \frac{4}{5} \end{bmatrix} \begin{bmatrix} \frac{4}{5} \\ \frac{3i}{5} \end{bmatrix}}{\begin{bmatrix} -\frac{3i}{5} & \frac{4}{5} \end{bmatrix} \begin{bmatrix} \frac{3i}{5} \\ \frac{4}{5} \end{bmatrix}}$$

$$= \frac{\hbar}{2} \frac{-\frac{12i}{25} + \frac{12i}{25}}{\frac{9}{25} + \frac{16}{25}}$$

$$= \frac{\hbar}{2} \frac{0}{1}$$

$$= 0$$

Calculate the expectation value of S_x^2.

$$\langle S_x^2 \rangle = \frac{\langle \chi \,|\, S_x^2 \,|\, \chi \rangle}{\langle \chi \,|\, \chi \rangle} = \frac{\chi^\dagger S_x^2 \chi}{1} = \begin{bmatrix} \frac{3i}{5} \\ \frac{4}{5} \end{bmatrix}^\dagger \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{3i}{5} \\ \frac{4}{5} \end{bmatrix}$$

$$= \frac{\hbar^2}{4} \begin{bmatrix} -\frac{3i}{5} & \frac{4}{5} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{3i}{5} \\ \frac{4}{5} \end{bmatrix}$$

$$= \frac{\hbar^2}{4} \begin{bmatrix} -\frac{3i}{5} & \frac{4}{5} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{4}{5} \\ \frac{3i}{5} \end{bmatrix}$$

$$= \frac{\hbar^2}{4} \begin{bmatrix} -\frac{3i}{5} & \frac{4}{5} \end{bmatrix} \begin{bmatrix} \frac{3i}{5} \\ \frac{4}{5} \end{bmatrix}$$

$$= \frac{\hbar^2}{4} \left(\frac{9}{25} + \frac{16}{25} \right)$$

$$= \frac{\hbar^2}{4}$$

Calculate the expectation value of S_y.

$$\langle S_y\rangle = \frac{\langle\chi\,|\,S_y\,|\,\chi\rangle}{\langle\chi\,|\,\chi\rangle} = \frac{\chi^\dagger S_y\chi}{\chi^\dagger\chi} = \frac{\begin{bmatrix}\frac{3i}{5}\\ \frac{4}{5}\end{bmatrix}^\dagger \frac{\hbar}{2}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\begin{bmatrix}\frac{3i}{5}\\ \frac{4}{5}\end{bmatrix}}{\begin{bmatrix}\frac{3i}{5}\\ \frac{4}{5}\end{bmatrix}^\dagger\begin{bmatrix}\frac{3i}{5}\\ \frac{4}{5}\end{bmatrix}} = \frac{\hbar}{2}\frac{\begin{bmatrix}-\frac{3i}{5} & \frac{4}{5}\end{bmatrix}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\begin{bmatrix}\frac{3i}{5}\\ \frac{4}{5}\end{bmatrix}}{\begin{bmatrix}-\frac{3i}{5} & \frac{4}{5}\end{bmatrix}\begin{bmatrix}\frac{3i}{5}\\ \frac{4}{5}\end{bmatrix}}$$

$$= \frac{\hbar}{2}\frac{\begin{bmatrix}-\frac{3i}{5} & \frac{4}{5}\end{bmatrix}\begin{bmatrix}-\frac{4i}{5}\\ -\frac{3}{5}\end{bmatrix}}{\begin{bmatrix}-\frac{3i}{5} & \frac{4}{5}\end{bmatrix}\begin{bmatrix}\frac{3i}{5}\\ \frac{4}{5}\end{bmatrix}}$$

$$= \frac{\hbar}{2}\frac{-\frac{12}{25}-\frac{12}{25}}{\frac{9}{25}+\frac{16}{25}}$$

$$= \frac{\hbar}{2}\frac{-\frac{24}{25}}{1}$$

$$= -\frac{12\hbar}{25}$$

Calculate the expectation value of S_y^2.

$$\langle S_y^2\rangle = \frac{\langle\chi\,|\,S_y^2\,|\,\chi\rangle}{\langle\chi\,|\,\chi\rangle} = \frac{\chi^\dagger S_y^2\chi}{1} = \begin{bmatrix}\frac{3i}{5}\\ \frac{4}{5}\end{bmatrix}^\dagger \frac{\hbar}{2}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\frac{\hbar}{2}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\begin{bmatrix}\frac{3i}{5}\\ \frac{4}{5}\end{bmatrix}$$

$$= \frac{\hbar^2}{4}\begin{bmatrix}-\frac{3i}{5} & \frac{4}{5}\end{bmatrix}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\begin{bmatrix}\frac{3i}{5}\\ \frac{4}{5}\end{bmatrix}$$

$$= \frac{\hbar^2}{4}\begin{bmatrix}-\frac{3i}{5} & \frac{4}{5}\end{bmatrix}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\begin{bmatrix}-\frac{4i}{5}\\ -\frac{3}{5}\end{bmatrix}$$

$$= \frac{\hbar^2}{4}\begin{bmatrix}-\frac{3i}{5} & \frac{4}{5}\end{bmatrix}\begin{bmatrix}\frac{3i}{5}\\ \frac{4}{5}\end{bmatrix}$$

$$= \frac{\hbar^2}{4}\left(\frac{9}{25}+\frac{16}{25}\right)$$

$$= \frac{\hbar^2}{4}$$

Calculate the expectation value of S_z.

$$\langle S_z\rangle = \frac{\langle \chi \,|\, S_z \,|\, \chi\rangle}{\langle \chi \,|\, \chi\rangle} = \frac{\chi^\dagger S_z \chi}{\chi^\dagger \chi} = \frac{\begin{bmatrix}\frac{3i}{5}\\ \frac{4}{5}\end{bmatrix}^\dagger \frac{\hbar}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\frac{3i}{5}\\ \frac{4}{5}\end{bmatrix}}{\begin{bmatrix}\frac{3i}{5}\\ \frac{4}{5}\end{bmatrix}^\dagger \begin{bmatrix}\frac{3i}{5}\\ \frac{4}{5}\end{bmatrix}} = \frac{\hbar}{2}\frac{\begin{bmatrix}-\frac{3i}{5} & \frac{4}{5}\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\frac{3i}{5}\\ \frac{4}{5}\end{bmatrix}}{\begin{bmatrix}-\frac{3i}{5} & \frac{4}{5}\end{bmatrix}\begin{bmatrix}\frac{3i}{5}\\ \frac{4}{5}\end{bmatrix}}$$

$$= \frac{\hbar}{2}\frac{\begin{bmatrix}-\frac{3i}{5} & \frac{4}{5}\end{bmatrix}\begin{bmatrix}\frac{3i}{5}\\ -\frac{4}{5}\end{bmatrix}}{\begin{bmatrix}-\frac{3i}{5} & \frac{4}{5}\end{bmatrix}\begin{bmatrix}\frac{3i}{5}\\ \frac{4}{5}\end{bmatrix}}$$

$$= \frac{\hbar}{2}\frac{\frac{9}{25} - \frac{16}{25}}{\frac{9}{25} + \frac{16}{25}}$$

$$= \frac{\hbar}{2}\frac{-\frac{7}{25}}{1}$$

$$= -\frac{7\hbar}{50}$$

Calculate the expectation value of S_z^2.

$$\langle S_z^2\rangle = \frac{\langle \chi \,|\, S_z^2 \,|\, \chi\rangle}{\langle \chi \,|\, \chi\rangle} = \frac{\chi^\dagger S_z^2 \chi}{1} = \begin{bmatrix}\frac{3i}{5}\\ \frac{4}{5}\end{bmatrix}^\dagger \frac{\hbar}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\frac{\hbar}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\frac{3i}{5}\\ \frac{4}{5}\end{bmatrix}$$

$$= \frac{\hbar^2}{4}\begin{bmatrix}-\frac{3i}{5} & \frac{4}{5}\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\frac{3i}{5}\\ \frac{4}{5}\end{bmatrix}$$

$$= \frac{\hbar^2}{4}\begin{bmatrix}-\frac{3i}{5} & \frac{4}{5}\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\frac{3i}{5}\\ -\frac{4}{5}\end{bmatrix}$$

$$= \frac{\hbar^2}{4}\begin{bmatrix}-\frac{3i}{5} & \frac{4}{5}\end{bmatrix}\begin{bmatrix}\frac{3i}{5}\\ \frac{4}{5}\end{bmatrix}$$

$$= \frac{\hbar^2}{4}\left(\frac{9}{25} + \frac{16}{25}\right) = \frac{\hbar^2}{4}$$

The reason $\langle S_x^2\rangle = \langle S_y^2\rangle = \langle S_z^2\rangle = \hbar^2/4$ is because each Pauli matrix is its own inverse.

$$\langle S_j^2\rangle = \langle \chi \,|\, S_j^2 \,|\, \chi\rangle = \left\langle \chi \,\middle|\, \frac{\hbar}{2}\sigma_j \frac{\hbar}{2}\sigma_j \,\middle|\, \chi\right\rangle = \frac{\hbar^2}{4}\langle \chi \,|\, \sigma_j \sigma_j^{-1} \,|\, \chi\rangle = \frac{\hbar^2}{4}\langle \chi \,|\, \mathsf{I} \,|\, \chi\rangle = \frac{\hbar^2}{4}\langle \chi \,|\, \chi\rangle = \frac{\hbar^2}{4}(1) = \frac{\hbar^2}{4}$$

Part (c)

Calculate the uncertainties in S_x, S_y, and S_z.

$$\sigma_{S_x} = \sqrt{\langle S_x^2\rangle - \langle S_x\rangle^2} = \sqrt{\frac{\hbar^2}{4} - (0)^2} = \frac{\hbar}{2}$$

$$\sigma_{S_y} = \sqrt{\langle S_y^2\rangle - \langle S_y\rangle^2} = \sqrt{\frac{\hbar^2}{4} - \left(-\frac{12\hbar}{25}\right)^2} = \frac{7\hbar}{50}$$

$$\sigma_{S_z} = \sqrt{\langle S_z^2\rangle - \langle S_z\rangle^2} = \sqrt{\frac{\hbar^2}{4} - \left(-\frac{7\hbar}{50}\right)^2} = \frac{12\hbar}{25}$$

Part (d)

The three uncertainty principles for spin angular momentum are

$$\sigma_{S_x}\sigma_{S_y} \geq \frac{\hbar}{2}|\langle S_z\rangle| \qquad \text{and} \qquad \sigma_{S_y}\sigma_{S_z} \geq \frac{\hbar}{2}|\langle S_x\rangle| \qquad \text{and} \qquad \sigma_{S_z}\sigma_{S_x} \geq \frac{\hbar}{2}|\langle S_y\rangle|.$$

Check to see that they're all satisfied.

$$\left(\frac{\hbar}{2}\right)\left(\frac{7\hbar}{50}\right) \overset{?}{\geq} \frac{\hbar}{2}\left|-\frac{7\hbar}{50}\right| \qquad \left(\frac{7\hbar}{50}\right)\left(\frac{12\hbar}{25}\right) \overset{?}{\geq} \frac{\hbar}{2}|0| \qquad \left(\frac{12\hbar}{25}\right)\left(\frac{\hbar}{2}\right) \overset{?}{\geq} \frac{\hbar}{2}\left|-\frac{12\hbar}{25}\right|$$

$$\frac{7\hbar^2}{100} \geq \frac{7\hbar^2}{100} \qquad \frac{42\hbar^2}{625} \geq 0 \qquad \frac{6\hbar^2}{25} \geq \frac{6\hbar^2}{25}$$